

Matter-wave interferometry in periodic and quasi-periodic arrays

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Abstract

We calculate within a Bose-Hubbard tight-binding model the matter-wave flow driven by a constant force through a Bose-Einstein condensate of ^{87}Rb atoms in various types of quasi-onedimensional arrays of potential wells. Interference patterns are obtained when beam splitting is induced by creating energy minigaps either through period doubling or through quasi-periodicity governed by the Fibonacci series. The generation of such condensate modulations by means of optical-laser structures is also discussed.

Key words: Laser applications, Bose-Einstein condensates in periodic potentials
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1 Introduction

A Bose-Einstein condensate (BEC) is a gas in which a macroscopic number of massive particles reside in the same quantum state (see [1] for a review of work done on quasi-pure BEC's produced since 1995). Experiments aimed at revealing the coherence of a BEC have demonstrated its matter-wave properties. In particular, condensate interferometry can be realized by splitting a BEC into two parts with a definite phase relationship, these parts being then brought into overlap and interference as for an optical laser beam that has gone through a beam splitter. Coherent splitting of a BEC has been achieved by optically induced Bragg diffraction [2] and a number of ingenious methods have been devised to extract a collimated beam of atoms from a BEC (see *e.g.* [3]).

A quasi-onedimensional (1D) array of potential wells is created for an atomic BEC by the interference of two optical laser beams which counterpropagate along the z axis, say, and are superposed on a highly elongated magnetic trap. Such an optical lattice provides an almost ideal periodic potential and has allowed the study of Bloch and Josephson-like oscillations [4,5,6] and of the mechanisms by which decoherence arises as in the transition from a superfluid to a Mott-insulator state [7].

The 1D optical lattice can be modified by means of auxiliary laser beams [8]. In particular, its periodicity can be doubled by adding two beams that are rotated by angles of 60° and 120° with respect to the z axis. For a suitable choice of the phases the potential seen by the BEC atoms takes the shape

$$U(z) = U_0[\sin^2(kz) + \beta^2 \sin^2(kz/2)], \quad (1)$$

where U_0 is the potential well depth, β^2 is the relative energy difference between adjacent wells, and k is the laser wavenumber determining the distance d of adjacent wells as $d = \pi/k$. In solid-state terminology, the doubling of the period when $\beta^2 \neq 0$ causes the opening of a minigap in the energy spectrum as a function of β^2 . A BEC driven through such a lattice by a constant force is coherently split by a combination of Bragg diffraction and of tunnelling through the minigap. In steady state a prominent interference pattern is produced and can be observed by monitoring the outgoing particle flow [8].

In this work we show how a quasi-periodic Fibonacci array of potential wells could be created by optical means and evaluate its density-of-states structure to display a series of approximate minigaps. We then show that this structure, unlike a simple periodic structure but similarly to a period-doubled structure, leads to an interference pattern under steady-state drive of the BEC by a constant force. The model and the behaviours of periodic arrays are briefly reviewed in Sec. 2 and Sec. 3, respectively. The Fibonacci array is treated in

Sec. 4, while Sec. 5 offers some concluding remarks.

2 The model

We use a 1D tight-binding Hamiltonian for the BEC atoms and a Green's function approach to evaluate their linear transport coefficient through the array of potential wells [8]. The Bose-Hubbard Hamiltonian for N bosons distributed inside n_s wells is

$$H_I = \sum_{i=1}^{n_s} [E_i |i\rangle\langle i| + \gamma_i (|i\rangle\langle i+1| + |i+1\rangle\langle i|)] . \quad (2)$$

Here, the parameters E_i and γ_i represent site energies and hopping energies, respectively, and depend on the number of bosons in the well labelled by the index i . In a tight-binding scheme the 1D condensate wavefunction in the i -th well is a Wannier function for the bosons in the external potential and, according to the early work of Slater [9], can be written as a Gaussian function of longitudinal width σ_z ,

$$\phi_i(z) = \phi_i(z_i) \exp[-(z - z_i)^2 / (2\sigma_z^2)] . \quad (3)$$

The parameters entering the effective Hamiltonian are given by

$$E_i = \int dz \phi_i(z) \left[-\frac{\hbar^2 \nabla^2}{2m} + U(z) + \frac{1}{2} g_{bb} |\phi_i(z)|^2 - maz + \mathcal{C} \right] \phi_i(z) \quad (4)$$

for the site energies and by

$$\gamma_i = \int dz \phi_i(z) \left[-\frac{\hbar^2 \nabla^2}{2m} + U(z) + \frac{1}{2} g_{bb} |\phi_i(z)|^2 + \mathcal{C} \right] \phi_{i+1}(z) . \quad (5)$$

for the hopping energies. In Eqs. (4) and (5) m is the boson mass, $a = F/m$ is the acceleration due to a constant external force F acting on the bosons, g_{bb} is the effective 1D boson-boson interaction parameter, and \mathcal{C} is a constant accounting for transverse effects in a cigar-shaped trap (for the determination of the parameters see Ref. [8]). We remark that in a tight-binding approach nonlinear interaction effects enter the self-consistent calculation of the axial width σ_z , resulting in a broadening of the Gaussian function, and also modify the on-site energies and the hopping energies. This approach is justified in the case of weak boson-boson coupling as for a ^{87}Rb BEC, on which we focus in this paper, and should be improved in a strong-coupling situation as is met on the approach to a Feshbach resonance [10].

In the Green's function method the calculation of the transmittivity of bosonic matter waves through the array of potential wells does not require an ex-

explicit solution of the Hamiltonian (2). The array is reduced by a renormalization/decimation technique to a single “dimer” [11], to which an incoming lead and an outgoing lead are connected. The steady-state transport coefficient is inferred from the scattered wavefunction of the leads in the presence of the dimer.

Period-doubling of the array as described by the expression of $U(z)$ in Eq. (1) yields an interference pattern as a function of the ratio T_B/τ , where $T_B = \hbar k/m a$ is the period of Bloch oscillations and τ is the average time needed for tunnelling twice across the minigap. The array acts in this case as an interferometer for bosonic matter waves and we construct below its optical analog. In essence the minigap plays the role of a medium with large refractive index, across which an evanescent wave couples two layers that allow real wave propagation. In the case of a Fibonacci array, on the other hand, periodicity is lost but quasi-periodicity induces the opening of a number of rather sharp depressions in the density of states (“quasi-minigaps”). The resulting fragmentation of the spectral density modifies the interference pattern, but does not erase it.

3 Interference from period doubling

When $\beta^2 = 0$ the ideal infinite lattice generated by $U(z)$ has a single period and its low-energy spectrum is that of a one-band system, as is shown in the left panel of Fig. 1. Period doubling causes the opening of a minigap in the total density of states (DOS) (see right panel in Fig. 1). The calculation of the DOS has been carried out by recursive algorithms as those in Refs. [13,14] and full details will be given in a later publication. The energy width ΔE of the minigap is fixed by the energy difference $|E_i - E_{i+1}|$ between two adjacent sites and in the limit of an infinite lattice no states are present inside the minigap. In real systems as in the experiments at LENS (see for example [12]) the condensate occupies about 100-200 wells. As a result of finite-size effects the gap is not completely empty and the bosons can easily be transferred by tunnelling between the two sub-bands.

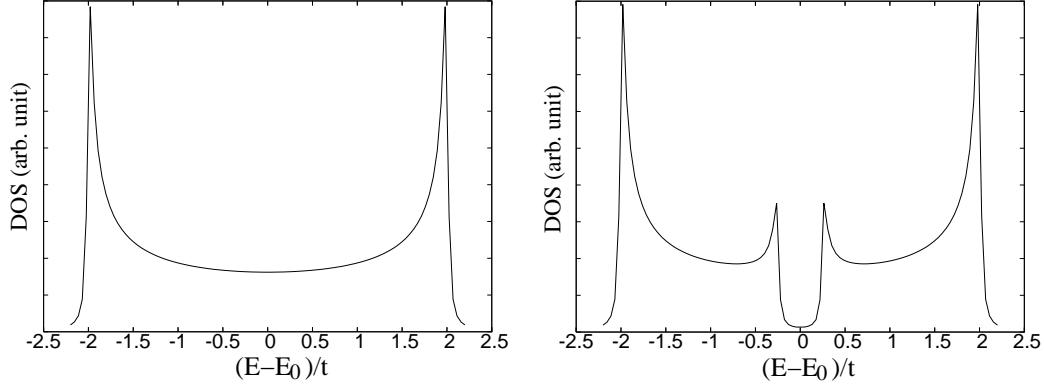


Fig. 1. Total DOS of a very long lattice (1000 sites) with a single period (left panel) and a doubled period (right panel), as a function of energy E referred to the central energy E_0 and with $4t$ being the total spectral width. In the single-period lattice E_0 is the site energy E_i and t the hopping energy γ_i , while in the double-period lattice $E_0 = (E_i + E_{i+1})/2$ and $t = [(E_i - E_{i+1})^2/2 + 4\gamma_i^2]^{1/2}/2$.

The addition of a potential maz causes a tilt of the bands in space and the density of states depends on both position and energy. Thus the condensed bosons are driven through the lattice and explore the whole band (in the single-period lattice) or both sub-bands (in the doubled-period lattice). On reaching the upper-energy state they are partly allowed to leave the lattice towards the continuum. For the evaluation of the number of transmitted particles we connect the system to incoming and outgoing leads, which mimic its coupling to the continuum by injecting and extracting a steady-state particle current. A schematic representation of the two-band situation is given in the top part of Fig. 2.

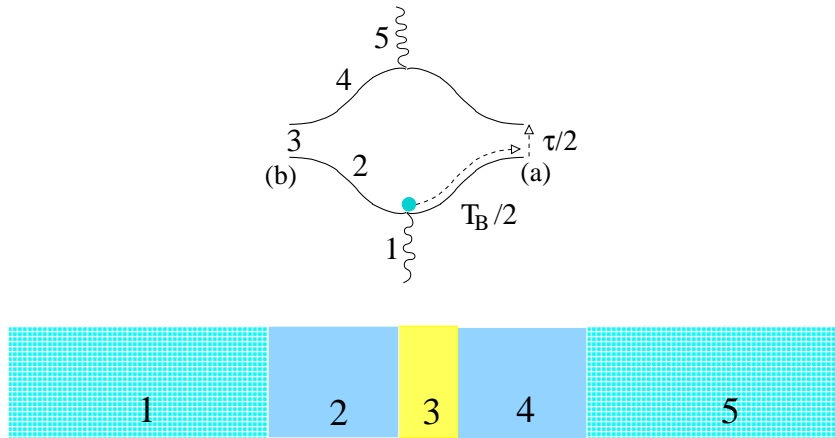


Fig. 2. Top: schematic representation of the two-band system connected to an incoming lead (1) and an outgoing lead (5), showing the half-period $T_B/2$ of Bloch oscillations and the tunnelling time $\tau/2$. Bottom: the equivalent birefringent system for light propagation.

The transmittivity of a BEC of ^{87}Rb atoms has been evaluated by using the scattering matrix formalism adapted to the case of out-of-equilibrium leads [8]. We have used the set of parameters $U_0 = 3.5 E_r$ with $E_r = \hbar^2 k^2 / 2m$, $\beta^2 \simeq 0.01$ and $k = 8.2 \mu\text{m}^{-1}$. In the single-period case the particle current varies monotonically with the external force and hence with the period of Bloch oscillations. After period doubling the boson wavepacket is split at the edge of the Brillouin zone (point (a) in Fig. 2), where it can either be Bragg-reflected to point (b) or tunnel into the upper sub-band. The interference between the wavepackets reaching point (b) by these various paths gives the outgoing current shown in Fig. 3. The minima in transmittivity towards the continuum are located at integer values of T_B/τ , where $\tau = (3\pi^2/8)(\hbar N/n_s \Delta E)$.

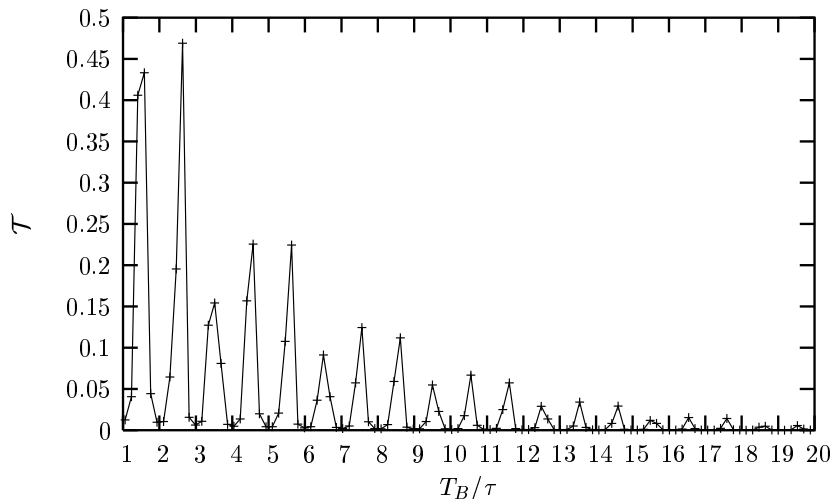


Fig. 3. Interference pattern in condensate transmittivity from period doubling, as a function of T_B/τ .

3.1 Optical equivalent

A condensate wavepacket propagating in an infinite doubled-period lattice is equivalent to a light beam of frequency ω travelling through a five-layer optical medium (see Fig. 2). The first and the last medium in the bottom part of Fig. 2 are semi-infinite and play the role of the two leads. The second and the fourth layer stand for the two energy bands, while the middle layer mimics the minigap.

Let $t_{i,j}$ and $r_{i,j}$ be the transmission and reflection coefficient at the interface between the media i and j , connected by $t_{i,j} = 1 + r_{i,j}$. For a suitable coupling between the leads and the lattice $t_{1,2}$ and $t_{4,5}$ have unitary modulus and on crossing these interfaces the wave takes up an irrelevant phase factor. At the 2-3 and 3-4 interfaces we can mimic the effect of the minigap in Bragg-reflecting

the BEC by imposing total reflection of light *via* $r_{2,3} = \exp(i\alpha_{2,3})$ and $r_{3,4} = \exp(i\alpha_{3,4})$, and allow for tunnel by propagation through layer 3 only *via* an evanescent wave. If the media 2 and 4 have the same refractive index and the same optical depth, we have $\alpha_{3,4} = \alpha_{2,3} - \pi$ and by symmetry we can set $\alpha_{2,3} = -\alpha_{3,4} = \pi/2$. We can then use the recursive formula [15]

$$\begin{cases} r_{i,j+2} = \frac{r_{i,j+1} + r_{j+1,j+2}e^{2i\alpha_{j+1}}}{1 + r_{i,j+1}r_{j+1,j+2}e^{2i\alpha_{j+1}}} \\ t_{i,j+2} = \frac{t_{i,j+1} + t_{j+1,j+2}e^{i\alpha_{j+1}}}{1 + r_{i,j+1}r_{j+1,j+2}e^{2i\alpha_{j+1}}} \end{cases} \quad (6)$$

to calculate the transmission coefficient between the media 1 and 5, where α_j is the phase shift acquired by light travelling through the j -th layer. To pursue the analogy with the doubled-period lattice we have set $\alpha_2 = \alpha_4 = \omega T_B/2$ and $\alpha_3 = i\omega\tau/2$, with $\tau = 2\pi/\omega$. The total transmission coefficient $|t_{1,5}|^2$ for the light intensity is then found to have minima when the ratio T_B/τ takes integer values, as in the case of the doubled-period lattice.

The main difference between the two patterns is that in the BEC case the height of the peaks is largest at low values of T_B/τ (see Fig. 3), whereas in the optical analog all peaks have the same height. Decreasing T_B leads to a decrease of the optical depth of media 2 and 4 in the five-layer system, while for the condensate it means that the bosons leave the lattice towards the continuum after having travelled through a lower number of sites. Therefore, decreasing T_B is equivalent in this case to shortening the array and hence helps the tunnelling.

4 Interference from a Fibonacci chain

The symmetry of an optical potential that is created by the interference of optical laser beams is completely determined by the geometric arrangement of the beams. Therefore, one can not only realize lattices with various symmetries in the laboratory, but also design quasiperiodic optical potentials. Here we give as an example the use of the projection method that takes origin from solid state physics for generating a Fibonacci chain (see left panel in Fig. 4). The 1D Fibonacci arrangement is obtained from a 2D periodic square lattice by projecting all sites belonging to a strip with the irrational slope $\alpha = \arctan(2/(\sqrt{5} + 1))$ onto a line with the same slope (see for instance [16]). With this construction the distance between two neighbouring projected sites can take two different values, A or B say. Their ratio $A/B = (\sqrt{5} + 1)/2$ is the so-called golden ratio and the sequence of distances follows the Fibonacci chain rule $ABBABABB \dots$. This sequence can be obtained by the transformation rule $A \rightarrow B$ and $B \rightarrow BA$.

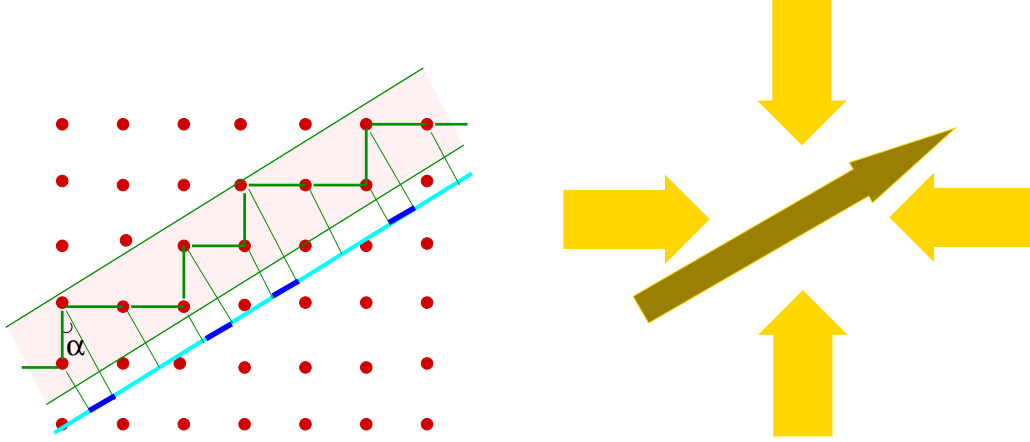


Fig. 4. Left panel: the projection method for creating the Fibonacci chain. Right panel: the equivalent set-up for a Fibonacci optical chain realized within five laser beams.

This method can be applied to an atomic gas by using a five-laser configuration as is shown in the right panel of Fig. 4. Four laser beams build the 2D square lattice by their interference. The fifth laser is aligned with the longitudinal axis of the cigar-shaped magnetic trap and drives the condensed bosons along the direction of the Fibonacci array. Within this model the hopping energies follow the Fibonacci sequence, but in order to make a direct comparison with the results on the periodic 1D lattices we consider below the case in which the site energies (rather than the bond lengths) form the Fibonacci chain. This scheme should at least qualitatively give the correct physical picture of matter waves propagating through a quasi-periodic array.

The total density of states for a very long Fibonacci array is shown in the left panel of Fig. 5. The fragmentation of the spectrum is typical of quasi-periodic and aperiodic systems, as it is well known from previous solid state studies (see for instance Ref. [17]). In particular, in the classical case of a quasi-periodic Fibonacci chain the spectrum is known to be a Cantor set with measure zero. For a chain of 100 wells, the site-projected density of states (see right panel in Fig. 5) has a general envelope resembling that of the single-period lattice, but is modified by the quasi-periodicity favouring the population of certain sites.

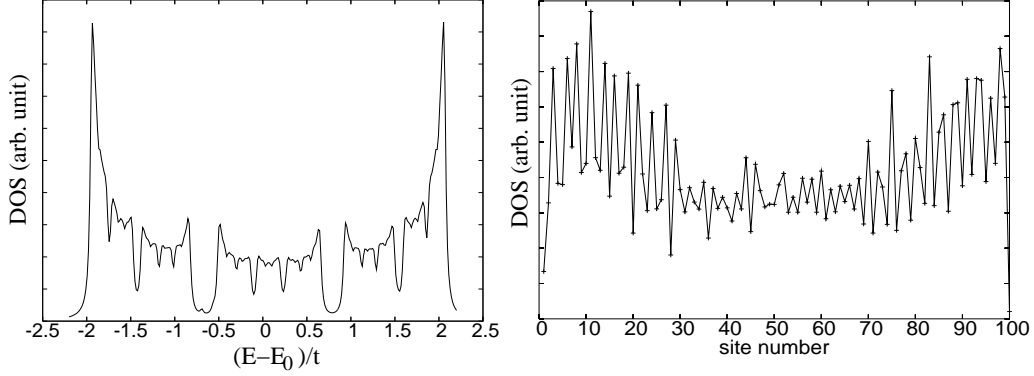


Fig. 5. Left panel: total DOS of a very long Fibonacci chain (1000 sites) as a function of $(E - E_0)/t$. Right panel: projected DOS for a BEC driven by a constant force through a Fibonacci chain of 100 sites at constant energy $E = E_0 - 2t$, as a function of the site number.

The condensed bosons travelling through the chain explore an energy spectrum which on average is rather more akin to the single-period band structure than to the doubled-period one. Nevertheless, the transmittivity through a Fibonacci chain presents an interference pattern (see Fig. 6). The presence of peaks is a signature of quasi-periodicity leading to “quasi-minigaps” in the energy spectrum.

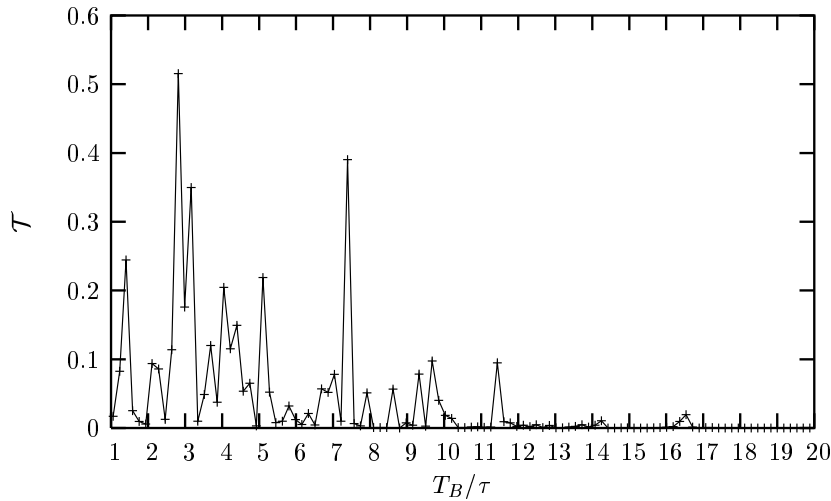


Fig. 6. Interference pattern in condensate transmittivity through a Fibonacci chain, as a function of T_B/τ .

5 Conclusions

In summary, we have reviewed earlier work on condensate transport in optical lattices and proposed an optical equivalent for the interference pattern that is generated by the opening of a minigap in the energy spectrum when matter waves are propagating in a lattice with doubled period.

We have then shown that condensate interference also results from the opening of sharp depressions in the spectral density of states for a matter wave propagating in a quasi-periodic array. Although quasi-periodicity is often said to be in some sense intermediate between perfect periodicity and complete disorder, it has been shown in earlier work [8] that in neither of these two cases an interference pattern of any sort is found in the absence of minigaps. We have also proposed a method by which a quasi-periodic modulation of the Fibonacci type may be created by optical means for a condensate in an elongated magnetic trap.

Acknowledgements

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